5.3 Common Factors

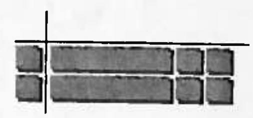
During a performance at a sea-life park, a dolphin jumps out of the water. Its height, h, in metres, above the water after t seconds can be approximated by the relation $h = 10x - 5x^2$. This relation can also be written as $h = 5x(2 - x)$, because the terms in the polynomial $10x - 5x^2$ have a common factor of $5x$.

Investigate A

Tools
- algebra tiles

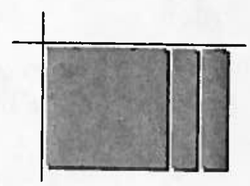
How can you use a model to find factors of a polynomial?

1. To factor $2x + 4$, use algebra tiles to create a rectangular area whose length and width represent the factors of the polynomial.
   a) Arrange two x-tiles and four unit tiles to form a rectangle with area $2x + 4$. Then, place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.
   b) Write an equation for the area as a product of the length and width.



2. Repeat step 1 for $6x + 18$. How many different rectangles can you find?

3. Use algebra tiles to find the factors of $x^2 + 2x$. Express the area as a product of the length and width.



4. Use algebra tiles to factor $2x^2 + 4x$. How many different rectangles can you find? Write an area statement for each one.

5. Use algebra tiles to factor each expression, if possible. If it is not possible, explain why.

a) $3x + 3$	b) $4x + 10$	c) $x^2 + 4x$
d) $2x^2 + 6x$	e) $2x + 5$	f) $4x^2 + 10x$

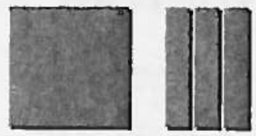
6. Reflect Explain how you can express a polynomial as a product of factors.

A polynomial is factored when it is written as a product of two or more polynomials. Factoring a polynomial is the reverse process of expanding. To factor a polynomial:

- Find the GCF of the terms.
- Write the GCF as the first factor outside a set of brackets.
- Divide each term by the GCF, writing the result inside the brackets.

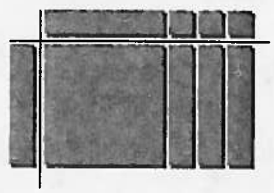
Example 1 Use a Model

Use algebra tiles to factor $x^2 + 3x$.



Solution

The polynomial $x^2 + 3x$ can be represented by a rectangle with area $x^2 + 3x$. The width of the rectangle is x and the length is $x + 3$. The dimensions of the rectangle are the factors of the polynomial.



$x^2 + 3x = x(x + 3)$

Example 2 Monomial Common Factor

Factor fully, if possible.

a) $6x + 3$
b) $8x^2 - 7x$
c) $25k^6 + 15k^4$
d) $21c^4d^3 - 28c^2d^5 + 7cd^3$
e) $5x^5y^3 + 7w^5z^2$

Solution

a) The GCF of the coefficients, 6 and 3, is 3.
There is no GCF of the variable parts.
Therefore, the GCF of the polynomial is 3.
Divide each term by 3.

$$6x + 3 = 3\left(\frac{6x}{3} + \frac{3}{3}\right)$$
$$= 3(2x + 1)$$

b) There is no common factor of the coefficients.
The GCF of the variable parts, x^2 and x, is x.
Therefore, the GCF of the polynomial is x.
Divide each term by x.

$$8x^2 - 7x = x\left(\frac{8x^2}{x} - \frac{7x}{x}\right)$$
$$= x(8x - 7)$$

c) The GCF of the coefficients, 25 and 15, is 5.
The GCF of the variable parts, k^6 and k^4, is k^4.
Therefore, the GCF of the polynomial is $5k^4$.
Divide each term by $5k^4$.

$$25k^6 + 15k^4 = 5k^4\left(\frac{25k^6}{5k^4} + \frac{15k^4}{5k^4}\right)$$
$$= 5k^4(5k^2 + 3)$$

When I compare the exponents, the GCF of the variable parts is the variable with the least exponent.

Apply the exponent laws.

d) The GCF of the coefficients, 21, −28, and 7, is 7.
The GCF of the variable parts, c^4d^3, c^2d^5 and cd^3, is cd^3.
Therefore, the GCF of the polynomial is $7cd^3$.
Divide each term by $7cd^3$ mentally.
$21c^4d^3 - 28c^2d^5 + 7cd^3 = 7cd^3(3c^3 - 4cd^2 + 1)$

I can check my answer by expanding.

$7cd^3(3c^3 - 4cd^2 + 1)$
$= 21c^4d^3 - 28c^2d^5 + 7cd^3$

e) Since the GCF of the terms of the polynomial $5x^5y^3 + 7w^5z^2$ is 1, it is not factorable.

Example 3 Binomial Common Factor

Factor.

a) $3x(y + 1) + 7z(y + 1)$

b) $2x(x - 3) - 5(x - 3)$

Solution

a) Think of $(y + 1)$ as one factor.
The GCF is the binomial $(y + 1)$.
Divide each term by $(y + 1)$ mentally.
$3x(y + 1) + 7z(y + 1)$
$= (y + 1)(3x + 7z)$

Write the GCF first. Collect the $(3x + 7z)$ in the second set of brackets.

b) Think of $(x - 3)$ as one factor.
The GCF is the binomial $(x - 3)$.
Divide each term by $(x - 3)$ mentally.
$2x(x - 3) - 5(x - 3)$
$= (x - 3)(2x - 5)$

Often there is no common factor for all the terms in a polynomial, but some of the terms have a common factor. A process of factoring by grouping can sometimes be used with these polynomials. This process involves factoring groups of terms first, instead of factoring the entire polynomial.

Example 4 Factor by Grouping

Factor.

a) $ax + ay + 2x + 2y$

b) $9x^2 + 15x + 3x + 5$

Solution

a) Group terms with a common factor. Factor the GCF from each grouping. Then, remove the binomial common factor.

$$\begin{aligned} &ax + ay + 2x + 2y \\ &= (ax + ay) + (2x + 2y) \\ &= a(x + y) + 2(x + y) \\ &= (x + y)(a + 2) \end{aligned} \qquad \text{or} \qquad \begin{aligned} &ax + ay + 2x + 2y \\ &= (ax + 2x) + (ay + 2y) \\ &= x(a + 2) + y(a + 2) \\ &= (a + 2)(x + y) \end{aligned}$$

b)

$$\begin{aligned} &9x^2 + 15x + 3x + 5 \\ &= (9x^2 + 15x) + (3x + 5) \\ &= 3x(3x + 5x) + 1(3x + 5) \\ &= (3x + 5)(3x + 1) \end{aligned} \qquad \text{or} \qquad \begin{aligned} &9x^2 + 15x + 3x + 5 \\ &= (9x^2 + 3x) + (15x + 5) \\ &= 3x(3x + 1) + 5(3x + 1) \\ &= (3x + 1)(3x + 5) \end{aligned}$$

Key Concepts

- Factoring a polynomial is the opposite of expanding a polynomial.

  $$\xrightarrow{\text{Factoring}}$$
  $$x^2 + 3x = x(x + 3)$$
  $$\xleftarrow{\text{Expanding}}$$

- To find the GCF of a polynomial, find the GCF of the coefficients, and then find the GCF of the variable parts.
- To factor a polynomial, remove the GCF as the first factor, and then divide each term by the GCF to obtain the second factor.
  $8x^2y^3 - 12x^4y = 4x^2y(2y^2 - 3x^2)$
- For polynomials with more than one variable, the GCF of the variable parts is the product of the common bases with the least exponent.
  The GCF of $2x^3y^4z^2 + 4x^2y^2z^3$ is $2x^2y^2z^2$.
- A common factor is not necessarily a monomial.
  $a(x + 2) + b(x + 2)$ has a binomial common factor of $(x + 2)$.
- To factor by grouping, factor groups of two terms with a common factor to produce a binomial common factor.

$$\begin{aligned} bx + 3x + by + 3y &= (bx + 3x) + (by + 3y) \\ &= x(b + 3) + y(b + 3) \\ &= (b + 3)(x + y) \end{aligned}$$